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INVESTIGATION OF THE WAYS PROSPECTIVE MATHEMATICS TEACHERS RESPOND TO STUDENTS' ERRORS: AN EXAMPLE OF THE EQUAL SIGN

(Research article)

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Abstract

This study aims to determine how prospective middle school mathematics teachers respond to students' errors in the questions about the equal sign. This study utilizes case study method. In this case study, hypothetical scenarios, involving three common error types related to the equal sign, have been prepared by using the possible examples of student work. Through these scenarios, one-to-one interviews were conducted with seven prospective middle school mathematics teachers. In line with the data obtained in these interviews, it was seen that the prospective teachers used seven different ways to respond to students' errors related to the equal sign: showing the error, showing the right solution, guiding to find the right answer, guiding to find the error, re-explaining the concept, in-depth research, and false intervention. In addition, it was determined that two prospective teachers intervened incorrectly by taking an approach that could support the thought that led to the error. In the light of the findings, it was seen that the prospective teachers had a limited understanding of the equal sign. This study suggests that mathematics educators should create appropriate learning opportunities to improve prospective teachers' understanding of the equal sign and their ability to respond to students' errors.

Keywords: equal sign, relational thinking, prospective teacher, giving feedback

Introduction

Teachers have the most important and direct impact on the quality of education. They are one of the most important elements of the education system due to their responsibilities in educational activities (MEB, 2017). While considering this critical role of the teacher in education and training activities, significant studies have been carried out on the types of knowledge that they should have (Shulman, 1986 &1987). Shulman (1986) stated that teachers should have three knowledge areas: subject knowledge, pedagogical content knowledge, and curriculum knowledge. In addition, Shulman (1987) defined seven basic knowledge categories that a teacher should have. These are content knowledge, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge, knowledge of learners and their characteristics, knowledge of educational context and knowledge of educational ends, purposes, and values as well as their philosophical and historical grounds. This classification of Shulman applies to all disciplines. Ball, Thames, and Phelps (2008) adapted the model put forward by Shulman to mathematics teaching. Ball et al. (2008) divided the model they named "Domains of Mathematical Knowledge for Teaching" into two basic components: subject matter knowledge and pedagogical content knowledge. They divided the subject matter knowledge component into three as common content knowledge, specialized content knowledge, and horizon content knowledge. Common content knowledge is mostly about solving a problem correctly. “0/7=0”, “finding a number between 1.1 and 1.11”, and “recognizing a common mistake made in any operation” are examples of common content knowledge. Other people who know and use mathematics also have this type of knowledge. Specialized content knowledge includes mastery of teaching-specific mathematical knowledge and skills. This type of knowledge is not needed outside of the teaching environment. Horizon content knowledge involves knowing how one subject relates to other subjects in mathematics. For instance, a classroom teacher should know the relationship between a mathematics topic taught in the first grade and a mathematics topic taught in the third grade. In this way, a basis can be formed for the subjects to be taught/learned later. The pedagogical content knowledge component is divided into three as knowledge of content and students, knowledge of content and teaching, and knowledge of content and curriculum (Ball et al., 2008). Knowledge of content and student includes knowing students’ mistakes, misconceptions, and their reasons as well as students’ thinking styles about a certain mathematics subject. Knowledge of content and teaching includes information such as explaining the teaching of a subject in a certain order, ordering the examples to be used according to a certain order, knowing the advantages and disadvantages of the representations to be used to explain the subject (Ball et al., 2008).

Ball et al. (2008) stated that during the analysis of a student’s error, teachers could detect the error using their specialized content knowledge or knowledge of content and student (p. 403). In addition, for the detailed analysis of students’ answers, Van es and Sherin (2002) and Jacobs, Lamb, and Philipp (2010) brought the “noticing skill” frameworks to the literature.

“Noticing” includes identifying important situations that occur in teaching environments, establishing a relationship between certain situations, learning and teaching principles, and making sense of the reasons why situations occur (van Es & Sherin, 2002). Van es and Sherin (2002) proposed a three-stage framework called as Learning to Notice. These stages are “(1) identifying what is important or noteworthy in a teaching situation, (2) using what one knows about the context to reason about a situation, (3) making connections between specific classroom events and broader principles of teaching and learning”. Another framework named as Professional Noticing of Children’s Mathematical Thinking was proposed by Jacobs et al. (2010). They defined the structure of the framework in three stages: "attending to students’ strategies, interpreting students’ understandings, and deciding how to respond on the basis of students’ understandings” (p. 169). It is seen that these two frameworks are widely used in studies related to noticing skills of teachers or prospective teachers (Doğan & Kılıç, 2019). In this current study, we only focus on the way prospective teachers respond to students’ errors.

Equal Sign

Kaput (1999) divided algebraic thinking into five. One of them is the meaningful use of symbols. The equal sign and variable symbols are particularly highlighted within the meaningful use of symbols. Kaput (2008) divided the development of algebraic thinking into three stages. The first stage consists of the generalization of arithmetic and quantitative reasoning. One of the first phase components is the meaning of the equal sign and relational thinking. The meaningful use of the equal sign has an important place in different classifications of algebraic thinking (Kaput, 1999; Kaput, 2008; Usiskin, 1988). A limited understanding of the equal sign is one of the major barriers to learning algebra. Almost all of the transformations on the equations require understanding that the equal sign represents a relationship (Carpenter, Franke, & Levi, 2003). The meaningful use of the equal sign and symbols is an important component of algebraic thinking and students' success in algebra topics (Barody & Ginsburg, 1982; Herscovics & Linchevski, 1994; Kaput, 1999; Oktaç, 2009).

Carpenter, Levi, Franke, and Zeringue (2005) define relational thinking about equality as the ability to examine the relationships between quantities using the basic properties of numbers and operations. It is stated that students with a relational understanding of the equal sign are more successful in transitioning from arithmetic to algebra (Knuth, Alibali, Weinberg, McNeil, & Stephens, 2005; Knuth, Stephens, McNeil, & Alibali, 2006). In studies on the equal sign, the aim was to determine students' understanding of the equal sign. It has been observed that students have two basic views on the equal sign, operational and relational (Alibali, Knuth, Hattikudur, McNeil, & Stephans, 2007; Barody & Ginsburg, 1982; Carpenter et al., 2003; Knuth et al., 2005; Matthews, Rittle-Johnson, McEldoon, & Roger, 2012; Stephans, Knuth, Blanton, Isler, Gardiner, Marum, 2013). It is seen that students with the operational view make five common mistakes regarding the equal sign (Alibali et al., 2007; Barody & Ginsburg, 1982; Carpenter et al., 2003; Knuth et al., 2005; Matthews et al., 2012; Stephans et al., 2013). (1) Students who have an operational view of the equal sign define the equal sign as "the sign that separates a problem and its answer", "the symbol used to show the result of the operations", or "the symbol with the result written on the right side" (Carpenter et al., 2003; Matthews et al., 2012; Stephans et al., 2013). (2) Expressions such as 8=8, 6+4=3+7 and 13=7+6 are meaningless to students with this view, or they accept them as wrong (Barody & Ginsburg, 1982). (3) Students with this view ignore the number 5 on the right side of the equation "8+4=☐+5" and write 12 instead of ☐. This mistake made by the students is called ″the answer comes after the equal sign″. (4) Students with this view solve the question 8+4=☐+5 as 8+4=12+5=17. This mistake made by the students is called "extending the problem″. 5) Students with this view solve the question 8+4=☐+5 as 8+4+5=17. This mistake made by the students is called "adding all the numbers" (Carpenter et al., 2003). Students who have a relational view about the equal sign define the equal sign as "denotes balance" or "the quantities on both sides of the equation are the same". Students with this view use two solutions for the question 8+4=☐+5. These are basic relational (or computational relational) and comparative relational (or structural relational) strategies. (Matthews et al., 2012; Stephans et al., 2013). In the question given above, the basic relational strategy is applied as ″8+4=12 and it should be ☐+5=12. Therefore, ☐=7". On the other hand, the comparative relational strategy is applied as "8+4=☐+5, the number 4 has increased by one to 5. For equality, the number 8 must decrease by one to 7. Therefore, ☐ =7" (Carpenter et al., 2003). Suppose the students do not use comparative relational strategy of the equal sign. In that case, they cannot answer the questions in the form of "Explain whether the expression "89 + 44 = 87 + 46" is true or not, without performing additions" (Matthews et al., 2012). Students' understanding of the equal sign is summarized in table 1 (Carpenter et al., 2003; Matthews et al., 2012; Stephans et al., 2013).

Table 1. *Levels of thinking about the equal sign*

Level Name	Explanation	Example
Level 1: Rigid Operational	It is the lowest level. Students at this level correctly answer the questions including	Students are successful in question types such as "5+4=?". However, they are unsuccessful in questions

	operations on the left side equal sign and the result on the right of the equal sign.	such as "8=8, 5=?+4, 5+4=?+3."
Level 2: Flexible Operational	Students at this level make the operational definition of the meaning of the equal sign (Equal sign indicates the result of an operation, etc.). In addition to the previous level, they also answer questions with the result on the left side of the equal sign and the operations on the right side.	Students are successful in question types such as "6+3=?, ?=5+4 ve 8=8". They are not successful in questions such as "6+4=?+2"
Level 3: Basic (or Computational) Relational	Students at this level can make a relational definition of the meaning of the equal sign. They answer the questions that have operations on both sides of the equation correctly by performing the relevant operations.	Their way of answering the question types such as "6+3=?+4" is as follows: since 6+3=9, it should be ?+4=9. Therefore, ?=5."
Level 4: Comparative (or Structural) Relational	It is the highest level. Students correctly answer questions that have operations on both sides of the equal sign by comparing terms.	Their way of answering the question types such as "6+3=?+4" is as follows: "The number 3 increased by one to become 4. The number 6 must decrease by 1 to achieve equality. Therefore, ?=5 ."

Giving Feedback on Students' Errors

Teachers, who shape the teaching process by taking the students' thoughts into account, have a positive effect on the students' learning (Kilpatrick et al., 2001). Recognizing the students' errors and responding to them appropriately is one of the main tasks of teachers in teaching mathematics (National Council Teacher of Mathematics (NCTM), 2000). Son and Sinclair (2010) and Son (2013) found that prospective mathematics teachers gave feedback to students' errors in two different categories: "show-tell" and "give-ask". In the context of mathematical modeling activities, Didiş, Erbaş, and Çetinkaya (2016) found that the intervention styles of prospective secondary mathematics teachers towards students' errors are collected in five categories as "question-asking (questioning), explaining the correct answer directly, hinting at the correct solution, showing/telling the error and not intervening the error. Doğan and Kılıç (2009) found that prospective teachers gave feedback to students' errors in five different ways: no attempt (or only telling their solutions are wrong), explanation, orientation, exploration, and elaboration. The first three codes mentioned above were named as correct answer-focused categories and the last two codes were named as mathematical understanding-focused categories by the researchers. (Doğan & Kılıç, 2019).

Falkner et al. (1999) and Carpenter et al. (2003) emphasized that teachers have an important role in the development of students' understanding of equal sign and relational thinking. However, studies have found that teachers and prospective teachers have a limited understanding of noticing the difficulties that students face regarding equal sign and equality (Asquith, Stephens, Knuth, & Alibali, 2007; Stephens, 2006). Asquith et al. (2007) investigated middle school mathematics teachers' knowledge of students' understanding of equal sign and equation concepts. In the study, it was seen that the teachers' estimations of students' understandings of the concept of equations and students' understandings largely overlapped. However, it was observed that teachers had deficiencies in predicting students' understanding of the concept of the equal sign. In addition, regarding students' misconceptions about the variable and equal sign, it has been observed that; teachers rarely define these misconceptions as obstacles to the solutions of the problems that require the application of these concepts. Vermeulen and Meyer (2017) investigated the 5th and 6th grade students' misconceptions about the equal sign and also the "Mathematics Knowledge for Teaching" of the mathematics teachers of these students. According to the results of the research, it has been determined that the students have various misconceptions about the equal sign. On the other hand, they stated that the knowledge and skills of mathematics teachers on teaching the equal sign, such as identifying, preventing and correcting students' misconceptions, were insufficient (Vermeulen & Meyer, 2017). Stephans (2006) stated that prospective primary school teacher was insufficient in noticing students' thoughts about the equal sign. Within the scope of this study, the aim is to examine the prospective mathematics teachers' methods of giving feedback to students' errors about the equal sign. In line with this purpose, the problem of the research has been presented as "How do the prospective middle school mathematics teachers give feedback to the students' errors about the equal sign?".

Method

This current study uses a case study design which is one of the qualitative research methods. The most basic feature of qualitative case studies is the in-depth investigation of one or more cases" (Yıldırım & Şimşek, 2006, p.77). In addition, Creswell and Poth (2018) explain case studies as a qualitative research method in which researchers investigates one or more situations in a certain period of time with the help of observations, interviews, documents, or reports. This study aims to deeply probe the prospective teachers' methods of giving feedback to the students' errors about the equal sign within different hypothetical scenarios. Due to this purpose, the case study was used in the study.

Participants

The study was carried out with 7 prospective teachers, 5 female and 2 male, studying in the 4th grade of the elementary mathematics teaching undergraduate program of a state university located in the north of Turkey. Convenience sampling was used to determine the prospective teachers who participated in the study. The prospective teachers were informed about the content of the study and this research was conducted with the prospective teachers who voluntarily agreed to participate in the study. The general academic grade point averages of the prospective teachers vary between 2.31 and 3.39. The prospective teachers were taught courses such as Analysis 1, Analysis 2, Linear Algebra etc. They took the mathematical method courses such as Special Teaching Methods 1, Special Teaching Methods 2, Misconceptions in Mathematics etc., in the undergraduate mathematics teaching program and were successful in these courses.

Development of Data Collection Tool

In the creation of the questions in the data collection tool, studies on the equal sign were examined (Alibali et al., 2007; Asquith et al., 2007; Barody & Ginsburg, 1982; Carpenter et al., 2003; Knuth et al., 2005; Matthews et al., 2012; Stephans et al., 2013). Three common errors made by students were identified from these studies. After these errors were identified, studies on the methods of intervention or feedback by teachers or prospective teachers to students' errors were examined (Didiş, Erbaş, & Çetinkaya, 2016; Didiş-Kabar, & Amaç, 2018; Doğan & Kılıç, 2019; Son & Sinclair, 2010; Son, 2013). According to these examinations, the teaching scenarios given in Figure 1, Figure 2 and Figure 3 were created in order to determine the methods of giving feedback by prospective teachers to the students' errors. It is seen that a similar method was used in the studies of Didiş-Kabar and Amaç (2018) and Son (2013). After the development of the data collection tool by the researchers, two experts in the mathematics education field investigated the tool. In line with their opinions and suggestions, the tool was updated, and therefore its final version was constructed.

A 7th-grade student named Fatih solved the question: "8+4=□+5, □=?" as below:

Fatih's solution: "since 8+4=12, thus □=12"

Examine Fatih's solution and answer the following question.

How would you give feedback to Fatih? Explain in detail how you would guide Fatih to get the right answer.

Figure 1. Hypothetical scenario for the error "The answer comes after the equal sign"

The studies of Carpenter et al. (2003), Didiş-Kabar and Amaç (2018), and Son (2013) were used to create the teaching scenario given in Figure 1. The error made in the hypothetical scenario in Figure 1 is due to attributing operational meaning to the equal sign. Here, the equal sign is thought of as the symbol with the operations on the left and the result on the right. Action is taken according to this idea. In this error type, the number 5 on the right of the equal sign is not taken into account.

A 7th-grade student named Yunus solved the question "if 8+4=□+5, □=?" as below:

Yunus's solution is: "8+4=12+5=17."

Examine Yunus's solution and answer the following question.

How would you give feedback to Yunus? Explain in detail how you would guide Yunus to the correct answer.

Figure 2. Hypothetical scenario for the error "Extending the problem (or continuing operations along a line)"

The studies of Carpenter et al., (2003), Didiş-Kabar and Amaç (2018), and Son (2013) were used to create the hypothetical scenario given in Figure 2. The error made in this teaching scenario stems from attributing an operational meaning to the equal sign rather than a relational one. This question is the same as the question given in Figure 1. However, the error made in solving the problem is different. The aspect that differs from the error given in Figure 1 is that the number "5" on the right side of the equal sign is also included in the solution, extending the problem or continuing the process along a line.

"Choose one of the options "correct", "incorrect" or "I don't know" for the following statement. Explain how you reached the conclusion.

8 = 8 Correct Incorrect I don't know"

"Eyüp's solution: "8=8 is incorrect because there is not an operation to conduct here"

Examine Eyüp's solution and answer the following question.

How would you give feedback to Eyüp? Explain in detail how you would guide Eyüp to get the right answer.

Figure 3. Hypothetical scenario for the "meaningless" error

The studies of Barody and Ginsburg (1982), Didiş-Kabar and Amaç (2018) and Son (2013) were used to create the hypothetical scenario given in Figure 3. The error made here arises from the thought that "the equal sign is used in cases where there are operations on at least one side of the equal sign". This error is due to attributing an operational meaning to the equal sign rather than a relational meaning. The difference of this question from the questions given in Figures 1 and 2 is that there is no operation to the right or left of the equal sign in the problem sentence.

Data Collection

Data were collected by semi-structured interviews. Before the interviews, prospective teachers were informed about the content of the study. It was explained that the data obtained during each of the interviews would be used for a scientific study and their personal information would not be shared anywhere. It is also stated that pseudonymous names will be used instead of actual names. After these briefings, an interview was held. The interviews were conducted online due to the pandemic conditions at the time they were held. In the interviews, the hypothetical scenario given in Figure 1 was shared with the prospective teachers, and after the opinions of the prospective teachers were taken, the hypothetical scenario given in Figure 2 was shared with the prospective teacher. After taking the opinions of the prospective teachers about the scenario given in Figure 2, the hypothetical scenario given in Figure 3 was shared with the prospective teachers. After taking the opinions of the prospective teachers about the scenario given in Figure 3, the interviews were concluded. Interviews were carried out individually and a separate interview was conducted with each prospective teacher. The duration of the interviews varied between 8-10 minutes.

Data Analysis

In this study, descriptive analysis was performed. The data obtained in this analysis method were summarized and interpreted according to the previously determined themes (Yıldırım & Şimşek, 2008, p. 224). The intervention or feedback methods of prospective teachers revealed in the studies Didiş, Erbaş and Çetinkaya (2016), Didiş-Kabar and Amaç (2018), Doğan and Kılıç (2019) and Son (2013) formed a framework for the data analysis of this study. The intervention methods of the prospective teachers in these studies were "notification, explanation, orientation, exploration, elaboration, non-intervention, telling/showing the error, explaining the correct answer, hinting at the correct solution, question-asking, explaining-showing, presenting information, making the student notice, moving the thought forward, re-explaining the subject/concept, creating cognitive conflict, researching student thinking in depth". Similar structures among these categories were combined under an existing or new category. The categories 'notification' and 'telling/showing the error' were merged under the name "showing the error". This category is based on actions such as "telling students their mistakes or mistakes in their solutions". The categories of 'explanation' or 'explaining the correct answer' were combined under the category of "showing the right solution". This category corresponds to the actions such as "explaining the correct solution to the students" The categories of 'orientation', 'hinting at the correct solution' and 'making the student notice his/her error' were handled as "guiding to find the right answer" and "guiding to find the error". These categories correspond to actions such as "making the students notice their mistakes or guiding them to the correct solution". The categories of 'presenting the information' and 're-explaining the subject/concept' were combined under the category of "re-explaining the concept". This category corresponds to the actions of "retelling the concepts/topics that the problem is related to". The categories of 'elaboration', 'moving the thought forward' and 'in-depth research on student thinking' were handled as "in-depth research". This category corresponds to the actions of "asking questions to reveal student thoughts and understanding". In addition, the category of "false intervention", which was not included in the studies above, was added to this study. This category corresponds to the actions of "lack of giving correct feedback on students' errors or taking an approach that can support the thought that led to the error". Therefore, the feedback methods used by the prospective teachers in this study were classified as "showing the error, showing the right solution, re-explaining the concept, guiding to find the right answer, guiding to find the error, in-depth research and false intervention" The coding process was done by two researchers independently within the framework of the above-mentioned categories. These codings were compared, and the differences that emerged were discussed, and the consensus was achieved.

Findings

It has been observed that the feedbacks given by the prospective teachers about the students' errors are grouped under seven categories. These categories and their explanations are given in Table 2.

Table 2. *The ways prospective teachers respond to students' errors*

Categories	Explanations
Showing the Error	Telling/showing students' errors in their solutions
Showing the Right Solution	Explaining/showing the correct solution to students
Guiding to Find the Right Answer	Guiding students to the correct answer using models or through various orienting questions
Guiding to Find the Error	Guiding students to identify the errors using models or through various orienting questions
In-depth Research	Asking questions to elicit student's thinking and understandings
Re-explaining the Concept	Retelling the concepts/topics that the problem relates to
False Intervention	Taking an approach that can support the thought that led to the error

The methods of the prospective teachers in giving feedback according to the scenarios are shown in Table 3.

Table 3**.** *Distribution of prospective teachers' methods of giving feedback to students' errors*

	The Answer Comes After the Equal Sign	Extending the Problem	Meaningless
Showing the Error	Hülya	Gamze	-
Showing the Right Solution	Cansu	Cansu, Gamze, Hülya	Gamze, Hülya, Kaan
Guiding to Find the Right Answer	Ayla, Esin, Gamze, Kaan, Semih	Ayla	-
Guiding to Find the Error	Cansu, Hülya	Cansu, Hülya, Kaan, Semih	-
In-depth Research	Hülya	-	-
Re-explaining the Concept	-	Esin, Kaan, Semih	Ayla, Semih
False Intervention	-	-	Cansu, Esin

In Table 3, it was observed that, one of the prospective teachers' feedback methods for students' errors, in-depth research is only used in the error of "the answer comes after the equal sign". In addition, the prospective teachers used the false intervention only in the "meaningless" error while they used the "showing the right solution" method in all three types of errors. In addition, from Table 3, it is understood that some prospective teachers use more than one feedback method for a type of error.

In Table 3, it is seen that Cansu and Hülya used more than one method in giving feedback to the error "The answer comes after the equal sign". In the interview with Cansu on this error, it was determined that she used the methods of "guiding to find the error" to make the student realize his mistake and "showing the right solution" to reach the correct answer. In this type of error, the prospective teacher named Hülya stated that she would "give feedback according to the success level of the student". She stated that if the student who made this mistake had a low level of success, she would use the method of showing the error, if he had a medium level of success, she would use the method of "guiding to find the error" and if he had a high level of success, she would use the method of "in-depth research".

From Table 3, it was seen that Cansu, Hülya, Kaan, Semih and Gamze used two different methods of giving feedback on the error of "extending the problem". In the interview with Cansu and Hülya, they stated that they would use the method of "guiding to find the error" in order for the student to realize his mistake and "showing the right solution" to reach the correct answer. Kaan and Semih stated that they would use the method of "guiding to find the error" in order for the student to realize his mistake and "re-explaining the concept" to reach the correct answer. Gamze, on the other hand, stated that she would use the method of "showing the error" to make the student realize his mistake and "showing the right solution" to get the right answer.

Showing the Error: In this feedback method, prospective teachers directly show the error made by the students. The prospective teacher named Hülya did not use the "showing the error" method alone. She also stated that she would use the methods of "guiding to find the right answer" and "in-depth research". Using this feedback method, Gamze stated that she would also use the "showing the right solution" method. Prospective teachers did not use the "showing the error" method alone. It is seen that they also use another feedback method. The interview with Gamze, who used the method of showing the error for the error "extending the problem", is as follows.

Researcher: …How would you give feedback to Yunus? Explain in detail how you would guide Yunus to the correct answer.

Gamze: …I would explain with the double-pan balance scale model. I would put a weight of 8 kilograms and then 4 kilograms on a pan on the scale. On the other pan, I would put a weight of 12 kilograms and 5 kilograms. When I do this, the student would see that the scale is out of balance and 17 kilograms is heavier. This way I would show his error. After he realizes his mistake, I would explain to him the following: If you remember, the right side and the left side of the equal sign must be the same, equal or equivalent. But what did you do? You added 8 to 4, and then you added 12 to 5. Yet what should it have been? The sum of 8 and 4 should equal the sum of 5 and the box on the other side. So, in this case, the box should have been replaced with 7, not 12.

When the first four sentences of the interview with Gamze are examined, it is understood that she tried to show his mistake with the scale model. In the continuation of the interview, it is understood that she explained the correct solution of the problem to the student. Therefore, it is seen that the computational relational strategy is used to reach the correct solution.

Showing the Right Solution: In this feedback method, prospective teachers show the students the correct solution of the problem. The interview with Kaan, who used to show the right solution in giving feedback to the "meaningless" error, is as follows.

Researcher: …How would you give feedback to Eyüp? Explain in detail how you would guide Eyüp to get the right answer.

Kaan: When giving feedback to Eyüp, I would tell him that the equal sign can be used without any operations. I would demonstrate this using an equal-arm scale. I would place 8 of the same objects on both pans of the scale. By stating that the scales are in balance, I would have shown that 8=8 is true.

When the interview with Kaan is investigated, it is seen that there was an attempt to explain the correct answer verbally or with the scale model.

Guiding to Find the Right Answer: In this feedback method, prospective teachers guide students by solving similar samples and short-answer questions in order to find the right solution of the problem. The interview with Semih, who used this method in responding to the error "the answer comes after the equal sign", is as follows.

Researcher: … How would you give feedback to Fatih? Explain in detail how you would guide Fatih to get the right answer.

Semih: … I would solve similar examples. With the examples I solved, he would get the idea that the right side and left side of the equal sign should be equal to each other. In this way, I would try to get him to find the right answer. As an alternative, I would use concrete material. I would utilize apples of equal weight. For the 8 + 4 operation on one of the pans of the scale, I would first place 8 apples and then 4 apples on the same pan. I would put 5 apples on the other pan of the scale. I would ask how many more apples I should put in this *pan for the scales to balance. I would try to get him to find the right answer with these operations.*

After examining the interview with Semih, it is seen that he tries to give directions to the student for him to reach the right solution. It is also seen that he uses similar sample solving and scale model methods to guide him to the correct solution. In the scale model, it is understood that the computational relational strategy is used for the student to reach the correct answer.

Guiding to Find the Error: In this method of giving feedback, prospective teachers orientate students by asking them to do the operation with the use of the scale model in order to find their errors, solving similar examples and asking questions in a way that creates a contradiction. This method was not used alone. Prospective teachers, who used this method, also used a different method to enable the students to reach the correct answer (see Table 3). The interview with Cansu, who used the method of “guiding to find the error” for the error of “extending the problem”, is as follows.

Researcher: ... How would you give feedback to Yunus? Explain in detail how you would guide Yunus to the correct answer.

Cansu: When giving feedback to Yunus for the expression “8 + 4 = 12 + 5 = 17”, I would consider the terms 8+4 and 17. “Is the equation 8 + 4 = 17 true?” I would ask him and try to make him realize his mistake. By doing this, I would have him find his error. After that, I would explain the correct solution of the problem.

When the interview with Cansu is analyzed, we find she asks questions in a way that creates a contradiction in order for the student to realize his error.

In-depth Research: In this feedback method, prospective teachers ask the students questions to their find the right solution of the problem. The questions asked by the prospective teachers were in the form of “Why did you do it in this way? Why did you think in this way? ...". The interview with Hülya, who used the in-depth research method in responding to the error of "the answer comes after the equal sign", is as follows.

Hülya: In order to lead Fatih to the correct answer, I would try to understand his way of thinking by asking him questions like: ‘Why did you think like that?’ or ‘Why did you do it like that?’. It would be better to follow a way according to the answers I received to the questions here. I would like the expressions corresponding to the given operation to be displayed with the scale model. I would ask if the scales were in balance. I would ask how much each of the pans weighed in total. Then I would ask him what needed to be done to balance the scale. By using these questions and similar questions based on the answers I would get from the student, I would try to ensure that he reached the correct answer.

When the interview with Hülya is examined, we find she tends to ask questions while giving feedback to the student. It is understood that there was an attempt to use the computational relational strategy for the questions about the scale model.

Re-explaining the Concept: In this feedback method, prospective teachers re-explain the concept that the student made an error in or the relevant parts of the topic that they think are not fully understood. The interview with Ayla, who used the method of re-explaining the concept to give feedback on the “meaningless” error, is as follows.

Ayla: Here, we can use the scale example or model. I would try to make the student understand that with the scale model, the equal sign can be used not only to show the result of operations but also in different situations. I would place the same number of identical *objects on both pans of the scale. I would explain that the scales are in balance and this is shown as 4=4, 5=5,...*

When the interview with Ayla is investigated, the findings show she tends to re-teach the content by using the scale model to reach the correct answer.

False Intervention: In this feedback method, a teaching or feedback approach is used to support the thought that causes the student to make mistakes. This method was encountered only in the type of error coded as “Meaningless”. The interviews with Esin and Cansu, who used the false intervention method in giving feedback to the student in this type of error, are as follows:

Researcher: …How would you give feedback to Eyüp? Explain in detail how you would guide Eyüp to get the right answer.

Cansu: I would first ask the student what the expressions 4 + 4 and 2 + 6 are equal to. From here, I would try to reach the answer (8=8). So, I would write 2 + 6 = 4 + 4, then go one line below and try to get 8=8.

Esin : … Is it okay if I do something like this to get the correct answer here? For example, I would write 3 + 5 = 4 + 4 and use the scale model to show that they are equal. I would put 3- and 5-kilogram weights on one pan of the scale and two 4-kilogram weights on the other pan. After showing that the scale is in balance, I would ask him to add 3+5 and 4+4 separately. By doing this, I would show that 8=8.

The main reason of this error type is the thought that the equal sign can only be used in cases where there is an operation. Examining the interviews with Cansu and Esin, it is obvious that the statements made by these prospective teachers are mathematically correct. They seem to use addition operations to explain that 8=8 is true. However, the student does not accept the statement 8=8 since ‘there is no operation’. It is incorrect to give feedback to the student named Eyüp by using operations because it can support the idea that the equal sign can only be used in cases where there is an operation. The possible reasons why prospective teachers give incorrect feedback in this way can be shown as the inability to identify the source of the mistake made by the student correctly and the insufficient content knowledge about the equal sign.

Discussion and Conclusion

In the light of the findings, we observed that prospective teachers have different approaches to students’ errors regarding the equal sign. It was determined that prospective teachers gave feedback in the form of showing the error, showing the right solution, guiding to find the right answer, guiding to find the error, re-explaining the concept, false intervention and in-depth research. The result of this study is similar to the results of similar studies in the literature (Didis, Erbaş, & Çetinkaya, 2016; Doğan & Kılıç 2019; Son, 2013).

Within the scope of this study, it has been detected that some prospective teachers tend to ask questions for various purposes, such as understanding what students think about their errors, making students realize their errors by creating contradictions, or making them realize the right solution. It shows that prospective teachers who use such questions tend to use the “give and ask” approach expressed in the studies of Son and Sinclair (2010) and Son (2013). The purpose of using this approach is to enable students to be aware of their errors or reach the right solution instead of giving information directly. The questions used by the prospective teachers were mainly aimed at making the students realize their mistakes or finding a solution. In the studies conducted by Didiş et al. (2016), Doğan and Kılıç (2019), and Didiş-Kabar and Amaç (2018), the scholars obtained similar results. This finding can be explained by the fact that prospective teachers have a certain competence in considering students' solutions.

Within the context of this study, we observed that some prospective teachers used the intervention methods of showing the right solution, showing the error, and re-explaining the concept for the students' errors. This result reveals that prospective teachers tend to use the "show-tell" approach expressed in the studies of Son and Sinclair (2010) and Son (2013). It has been found out prospective teachers who use this method prefer to present the information directly rather than listening to the students (Didiş et al., 2016; Doğan & Kılıç, 2019; Didiş- Kabar & Amaç, 2018; Son & Sinclair, 2010; Son 2013).

One of the striking results reached in the study is that two prospective teachers used the false intervention method in the type of error coded as "Meaningless". The expressions or models used by the prospective teachers in giving feedback to the students were mathematically correct, but they supported the thought that caused the student to make mistakes. In this error type, the expression "8=8" was evaluated as "false" by the student because 'there was no action to be taken'. Two prospective teachers for this type error stated they could give feedback by using the scale model or symbolic representation (3+5=4+4 or 2+6=4+4). The feedback given in this way, may support the idea that the equal sign can only be used in cases where addition or other operations are involved. Prospective teachers used an improper method because they could not identify the source of the error correctly or they had a limited understanding of equality. The fact that prospective teachers have a limited understanding of equality and fail to notice their students' misunderstandings is consistent with the results of similar studies in the literature (Asquith et al., 2007; Stephens, 2006).

Students' understanding of equal signs is divided into four levels, from the lowest to the highest, as rigid operational, flexible operational, computational relational and comparative relational (Carpenter et al., 2003; Matthews et al., 2012; Stephans et al., 2013). The errors used in this study stem from attributing operational meaning to the equal sign. Prospective teachers used methods suitable for the computational relational level to give feedback on the errors mentioned in the study. In other words, they tried to bring the idea that the sum of the numbers on the right side of the equal sign and the sum of the numbers on the left side of the equal sign should be equal to each other. None of prospective teachers gave feedback on the comparative relational level, which is the highest level. This can be explained by the fact that prospective teachers have a limited understanding of the equal sign.

In this study, the ways in which prospective teachers gave feedback to the errors about the equal sign were examined through the hypothetical scenarios. The limitation of this study is the use of scenarios that may not reflect all the prospective teachers' responses to students' errors since there are no teacher-student interactions as in real situations. Despite the stated limitation, this study gives important clues about prospective teachers' understanding of the equal sign and the methods of giving feedback to the errors regarding the equal sign. The current study shows that the prospective teachers' understanding of the equal sign and their knowledge about the methods of giving feedback to the students' errors should be improved. For this reason, these and similar deficiencies of prospective teachers during their undergraduate education should be eliminated as much as possible. To ensure this, applications can be performed about the methods of giving feedback to students' errors in the content of the mathematical method courses or field experiences taken by the prospective teachers. These practices can be in the form of prospective teachers having one-on-one interviews with students or watching videos prepared for students' errors. With these applications, it can be provided that prospective teachers both see their own deficiencies and have more detailed information about the students' mathematical thinking styles.

References

Alibali, M. W., Knuth, E. J., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2007). A longitudinal examination of middle school students' understanding of the equal sign and equivalent equations. Mathematical Thinking and Learning, 9, 221–247. doi:10.1080/10986060701360902

Asquith, P., Stephens, A., Knuth, E., & Alibali, M. (2007). Middle school mathematics teachers' knowledge of students' understanding of core algebraic concepts: Equal sign and variable. Mathematical Thinking and Learning, 9(3), 249–272.

Ball, D. B., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special?. Journal of Teacher Education, 59(5), 389-407.

Baroody, A. J., & Ginsburg, H. P. (1982). The effects of instruction on children's understanding of the "equals" sign. Paper presented at the annual meeting of the American Educational Research Association, New York, NY, (ERIC Document Reproduction Service No. ED214765

Creswell, J. W., & Poth, C. N.   (2018). Qualitative Inquiry and Research Design: Choosing among five approaches (4th ed.). Sage Publications.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). Thinking mathematically: Integrating arithmetic and algebra in elementary school. Portsmouth, NH: Heinemann.

Carpenter, T., Levi, L., Franke, M., & Zeringue, J. (2005). Algebra in elementary school: Developing relational thinking. Zentralblatt für Didactic der Mathematik. (International Reviews on Mathematics Education), ZDM, 37(1), 53–59.

Didiş, M.G., Erbaş, A.K., & Çetinkaya, B. (2016). Investigating prospective mathematics teachers' pedagogical approaches in response to students' errors in the context of mathematical modeling activities. Elementary Education Online, 15(4), 1367-1384.

Didiş Kabar, M. G., & Amaç, R. (2018). Investigating pre-service middle-school mathematics teachers' knowledge of student and instructional strategies: An algebra case. Abant Izzet Baysal University Journal of Faculty of Education, 18 (1), 157-185.

Doğan, O., & Kılıç, H. (2019). Mathematical opportunities: Noticing and acting. Education and Science, 44 (199), 1-19, Doi: 10.15390/EB.2019.7593

Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). Children's understanding of equality: A foundation for algebra. Teaching Children Mathematics, 6(4), 56–60.

Herscovics, N., & Linchevski, L. (1994). A cognitive gap between arithmetic and algebra. Educational Studies in Mathematics, 27, 59–78.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. Journal for Research in Mathematics Education, 41(2), 169-202.

Kaput, J.J. (1999). Teaching and learning a new algebra. In. E. Fennema & T.A. Romberg (Eds.), Mathematics classrooms that promote understanding (pp.133-135). Mahwah, NJ: Erlbaum.

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In Kaput, J. J., Carraher, D. W., & Blanton, M. L. (Eds.), Algebra in the Early Grades (pp. 5–17). New York: Routledge, Taylor & Francis Group.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). Adding it up: Helping children learn mathematics. Washington, DC: National Academy Press.

Knuth, E., Alibali, M., McNeil, N., Weinberg, A., & Stephens, A. (2005). Middle school students' understanding of core algebraic concepts: Equivalence and variable. ZDM, 37(1), 68–76.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. (2006). Does understanding of the equal sign matter? Evidence from solving equations. Journal for Research in Mathematics Education, 37, 297–312.

Matthews, P., Rittle-Johnson, B., McEldoon, K., & Roger, T. (2012). Measure for Measure: What Combining Diverse Measures Reveals about Children's Understanding of the Equal Sign as an Indicator of Mathematical Equality, Journal for Research in Mathematics Education, 43 (3), 316- 350

Milli Eğitim Bakanlığı (MEB), (2017). Öğretmenlik mesleği genel yeterlikleri. Ankara: Öğretmen yetiştirme ve geliştirme genel müdürlüğü, https://oygm.meb.gov.tr/www/ogretmenlik-meslegi-genel-yeterlikleri/icerik/39 adresinden 16.01.2021 tarihinde 01.13'de indirilmiştir.

National Council of Teachers of Mathematics (NCTM). (2000). Principles and standards for school mathematics. Reston, VA.

Oktaç, A. (2009). Birinci dereceden tek bilinmeyenli denklemler ile ilgili kavram yanılgıları. (Editör: Erhan Bingölbali ve Mehmet Fatih Özmantar) İlköğretimde Karşılaşılan Matematiksel Zorluklar ve Çözüm Önerileri, sayfa 241-262, Ankara: Pegem Akademi,

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. Educational Researcher, 15(2), 4-14.

Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. Harvard Educational Review, 57(1), 1-21.

Son, J. W., & Sinclair, N. (2010). How preservice teachers interpret and respond to student geometric errors. School Science and Mathematics, 110(1), 31–46.

Son, J. W. (2013). How preservice teachers interpret and respond to student errors: Ratio and proportion in similar rectangles. Educational Studies in Mathematics, 84(1), 49–70.

Stephens, A. (2006). Equivalence and relational thinking: Preservice elementary teachers awareness of opportunities and misconceptions. Journal of Mathematics Teacher Education, 9, 246–278.

Stephens, A.C., Knuth, E. J., Blanton, M.L., Isler I, Gardiner, A.M., & Marum, T (2013). Equation structure and the meaning of the equal sign: The impact of task selection in eliciting elementary students' understandings. Journal of Mathematical Behavior, 32, 173– 182

Usiskin, Z.: 1988, 'Children's difficulties in beginning algebra', in The Ideas of Algebra, K-12, 1988 NCTM Yearbook, National Council of Teachers of Mathematics: Reston, Virginia, pp. 8-20.

van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. Journal of Technology and Teacher Education, 10, 571-576.

Vermeulen, C., & Meyer, B. (2017). The Equal Sign: Teachers' Knowledge and Students' Misconceptions, African Journal of Research in Mathematics, Science and Technology Education, 21:2, 136-147, DOI: 10.1080/18117295.2017.1321343

Yıldırım, A., & Şimşek, H. (2008). Sosyal Bilimlerde Nitel Araştırma Yöntemleri, Seçkin Yayıncılık, Ankara, 7. Baskı